The Concept of the Derivative

- The Numerical Approach – Difference Quotients
- The Graphical Approach – Slope from graphs
- The Analytic Approach

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DEFINITION Derivative

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{1}$$

provided the limit exists.

EXAMPLE 1 Applying the Definition

Differentiate (that is, find the derivative of) $f(x) = x^3$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= 3x^2$$

equation to find the slope of the tangent or slope of the curve or instaneous rate at any x.

DEFINITION (ALTERNATE) **Derivative at a Point**

The **derivative** of the function **f at the point $x = a$** is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \quad (2)$$

provided the limit exists.

means $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ when $x = a$

EXAMPLE 2 **Applying the Alternate Definition**

Differentiate $f(x) = \sqrt{x}$ using the alternate definition.

$$f'(a) = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \frac{(\sqrt{x} + \sqrt{a})}{(\sqrt{x} + \sqrt{a})}$$

$$= \lim_{x \to a} \frac{\cancel{x - a}}{\cancel{(x-a)}(\sqrt{x} + \sqrt{a})}$$

$$= \frac{1}{\sqrt{a} + \sqrt{a}}$$

$$= \frac{1}{2\sqrt{a}}$$

Notation

There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief, but does not name the independent variable.
$\dfrac{dy}{dx}$	"$dy\ dx$" or "the derivative of y with respect to x"	Names both variables and uses d for derivative.
$\dfrac{df}{dx}$	"$df\ dx$" or "the derivative of f with respect to x"	Emphasizes the function's name.
$\dfrac{d}{dx}f(x)$	"$d\ dx$ of f at x" or "the derivative of f at x"	Emphasizes the idea that differentiation is an operation performed on f.

Graphical Interpretation of Derivative: Slope of a Tangent Line

- The derivative of a function at a point equals the slope of the tangent line to the graph of the function at that point. Both equal the instantaneous rate of change.

Relationships between the Graphs of f and f'

1. Open Sketchpad and under Graph and Grid Form choose Rectangular Grid.

2. Plot the function $y = x^3$

3. Find the slope of the curve at $x = -2$, $x = -1$, $x = 0$, $x = 1$, and $x = 2$

4. Find y using your slope equation and plot those 5 points on your graph.

5. Highlight the function $y = x^3$ and under Number choose Define Derivative Function.

6. Highlight the Derivative Function and under Graph choose Plot Function.

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EXAMPLE 4 **Graphing f from f'**

Sketch the graph of a function f that has the following properties:

i. $f(0) = 0$;

ii. the graph of f', the derivative of f, is as shown in Figure 3.4;

iii. f is continuous for all x.

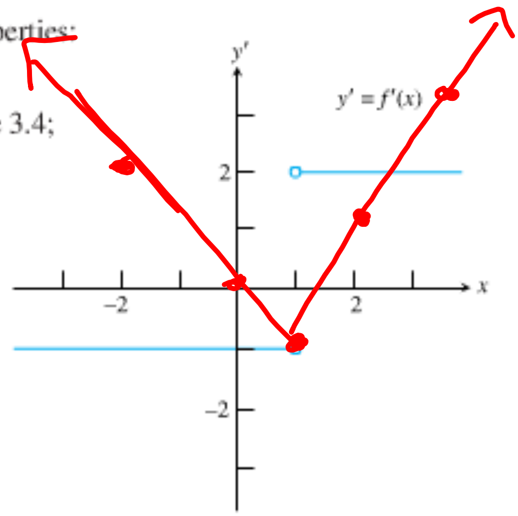

Figure 3.4 The graph of the derivative. (Example 4)

Relationship between a Function and Its Derivative Lab

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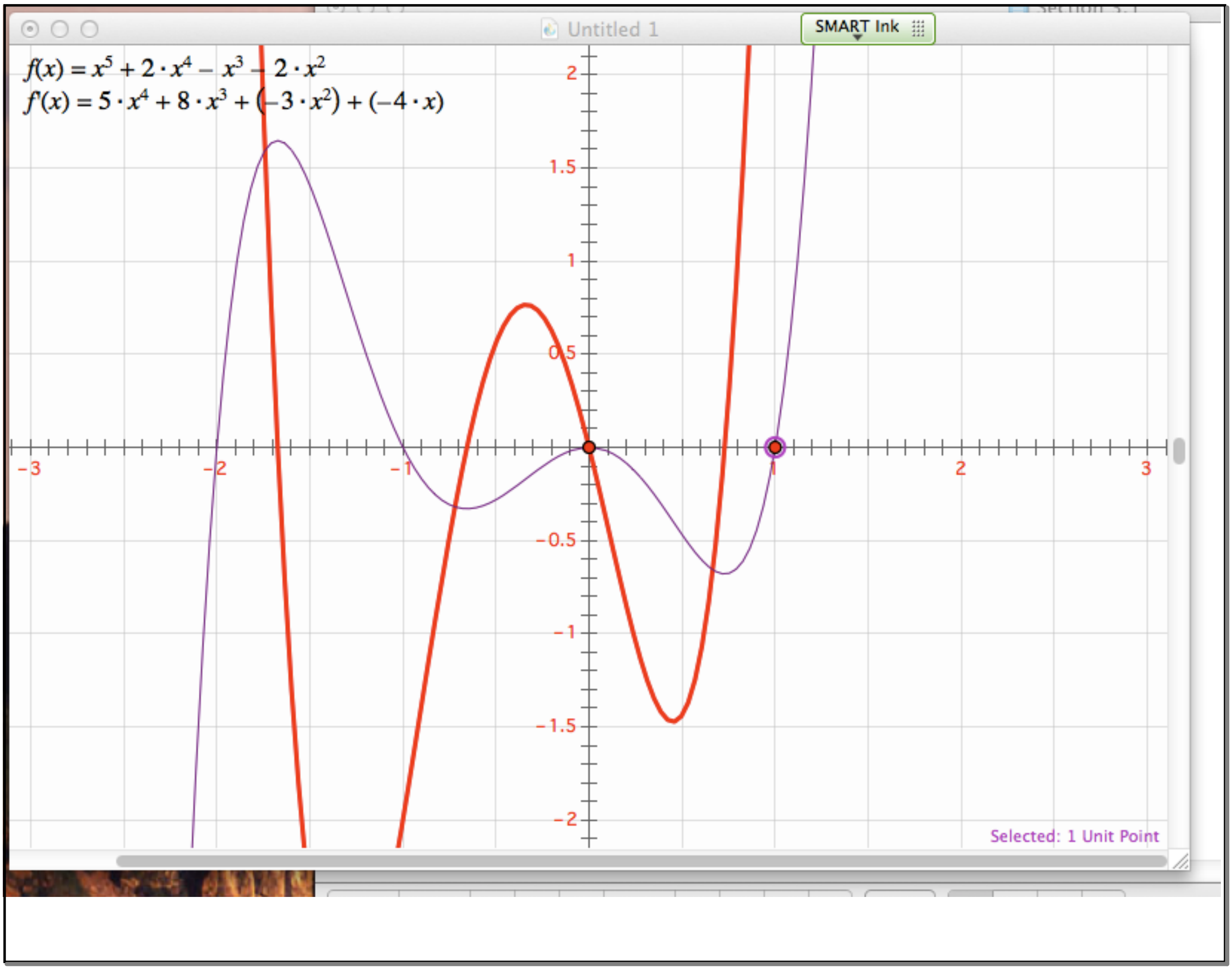

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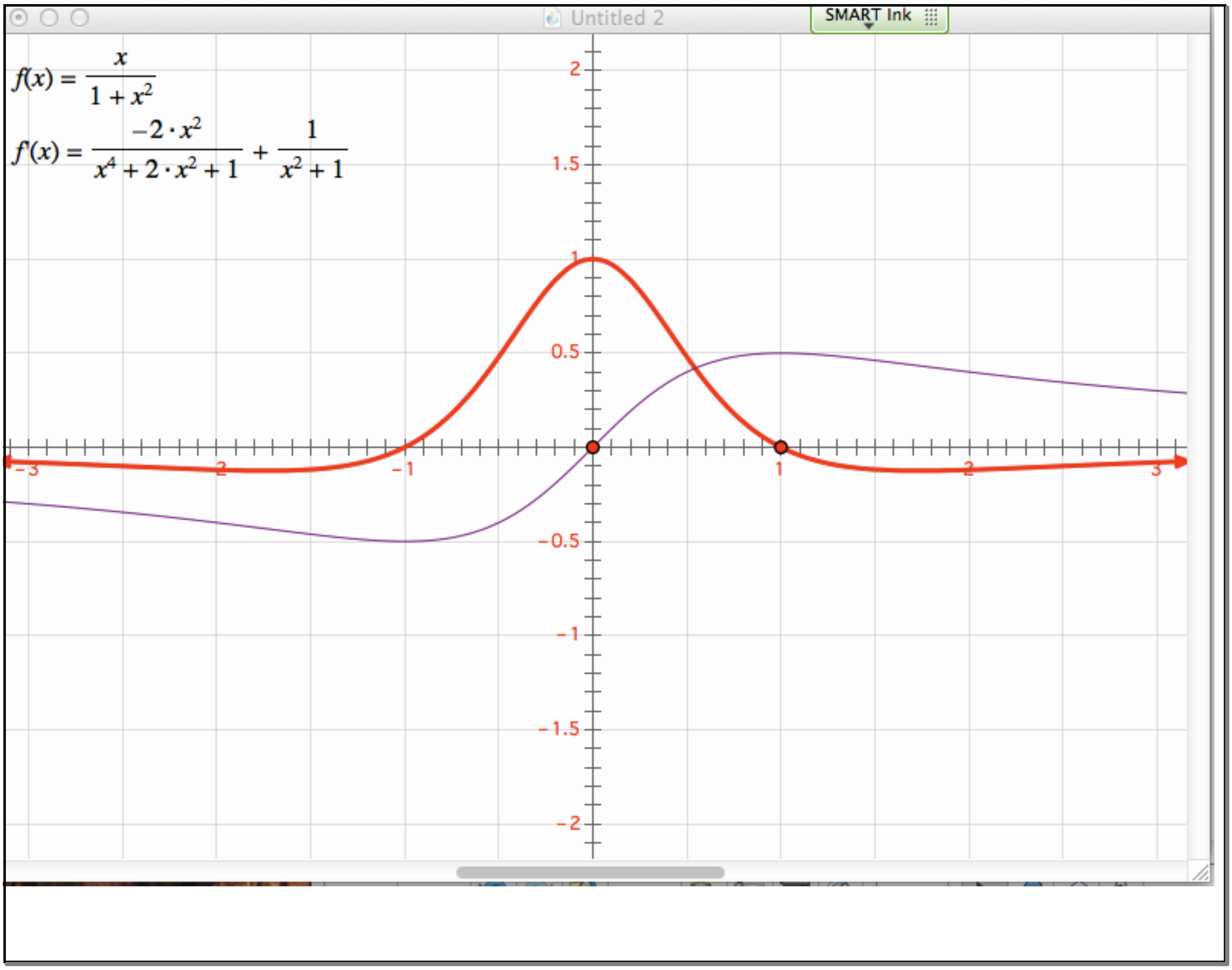

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EXPLORATION 1 Reading the Graphs

Suppose that the function f in Figure 3.3a represents the depth y (in inches) of water in a ditch alongside a dirt road as a function of time x (in days). How would you answer the following questions?

1. What does the graph in Figure 3.3b represent? What units would you use along the y'-axis?
2. Describe as carefully as you can what happened to the water in the ditch over the course of the 7-day period.
3. Can you describe the weather during the 7 days? When was it the wettest? When was it the driest?
4. How does the graph of the derivative help in finding when the weather was wettest or driest?
5. Interpret the significance of point C in terms of the water in the ditch. How does the significance of point C' reflect that in terms of rate of change?
6. It is tempting to say that it rains right up until the beginning of the second day, but that overlooks a fact about rainwater that is important in flood control. Explain.

Construct your own "real-world" scenario for the function in Example 3, and pose a similar set of questions that could be answered by considering the two graphs in Figure 3.3.

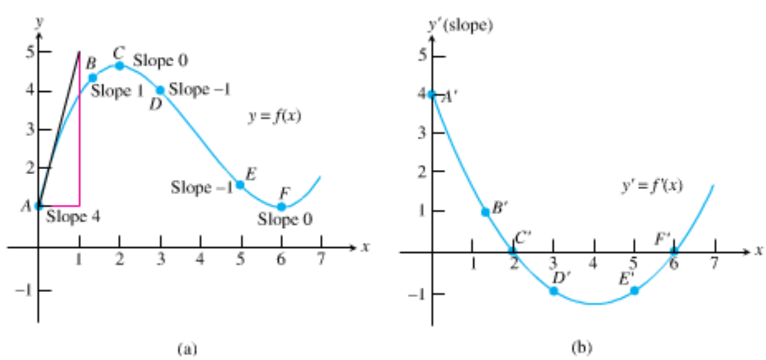

Figure 3.3 By plotting the slopes at points on the graph of $y = f(x)$, we obtain a graph of $y' = f'(x)$. The slope at point A of the graph of f in part (a) is the y-coordinate of point A' on the graph of f' in part (b), and so on. (Example 3)

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EXAMPLE 5 Estimating the Probability of Shared Birthdays

Suppose 30 people are in a room. What is the probability that two of them share the same birthday? Ignore the year of birth.

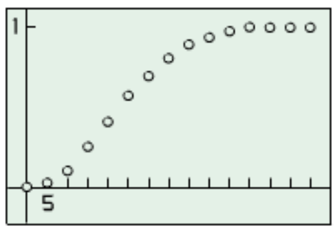
Table 3.1 Probabilities of Shared Birthdays

People in Room (x)	Probability (y)
0	0
5	0.027
10	0.117
15	0.253
20	0.411
25	0.569
30	0.706
35	0.814
40	0.891
45	0.941
50	0.970
55	0.986
60	0.994
65	0.998
70	0.999

Table 3.2 Estimates of Slopes on the Probability Curve

Midpoint of Interval (x)	Change (slope $\Delta y/\Delta x$)

Notice that the probabilities grow slowly at first, then faster, then much more slowly past $x = 45$. At which x are they growing the fastest? To answer the question, we need the graph of the derivative.



[–5, 75] by [–0.2, 1.1]

#6 Relationships between a function and its derivative Lab

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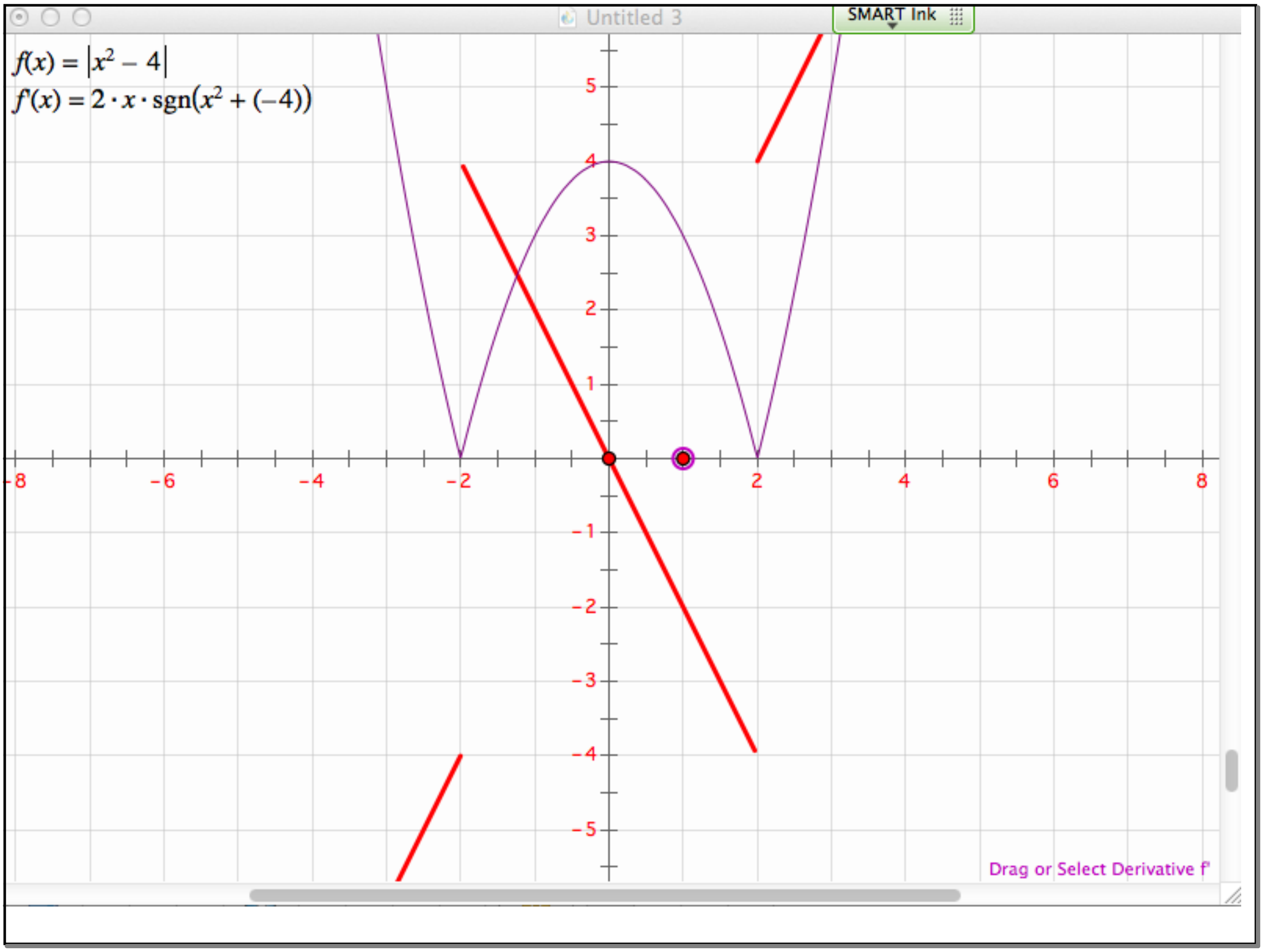
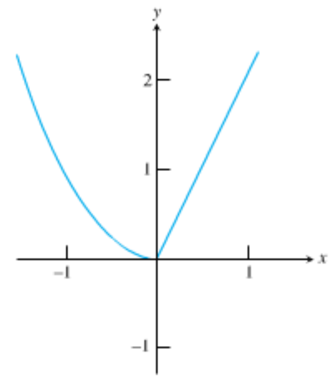


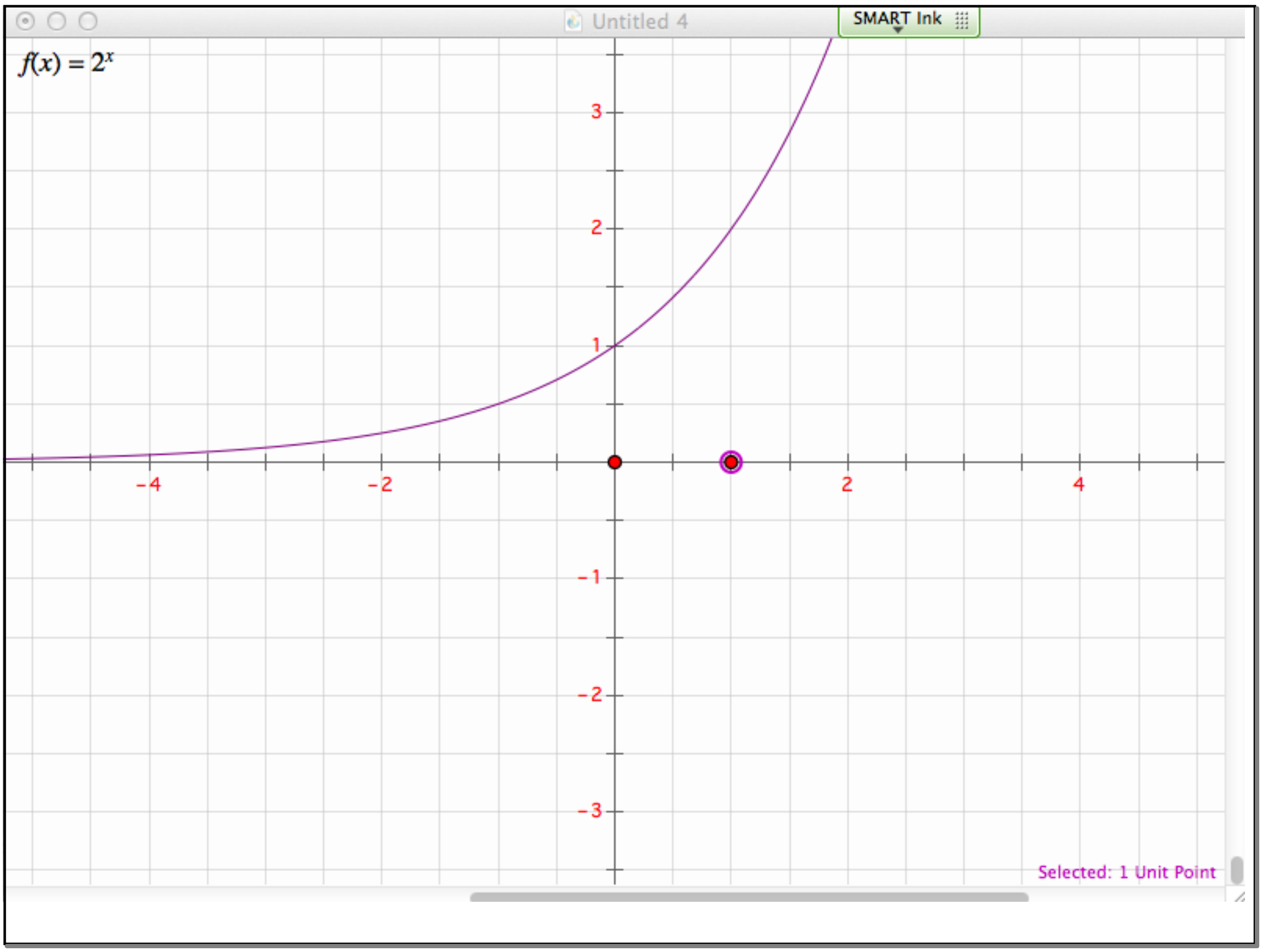
EXAMPLE 6 One-Sided Derivatives can Differ at a Point

Show that the following function has left-hand and right-hand derivatives at $x = 0$, but no derivative there (Figure 3.9).

$$y = \begin{cases} x^2, & x \leq 0 \\ 2x, & x > 0 \end{cases}$$



#7 Relationships between a function and its derivative lab


Untitled 4
SMART Ink
$f(x) = 2^x$
3
2
1
-4
-2
2
4
-1
-2
-3
Selected: 1 Unit Point